Letters

Alternate Forms of the Generalized Composite Scattering Matrix

P. L. OVERFELT AND D. J. WHITE

The cascading of two scattering matrices to form a composite scattering matrix is a well-known operation. Its advantage over the more straightforward cascading of transmission matrices lies in the fact that scattering matrices avoid the often occurring large transmission matrix elements that lead to computational difficulties. A fairly recent paper by Chu and Itoh [1] used direct substitution between inputs and outputs to derive a form for the generalized composite scattering matrix. We would like to point out that this is by no means the only method of derivation. "Star" matrix multiplication [2], [3], conversion of a generalized composite transmission matrix to its scattering equivalent [4], and signal flow graph analysis [5] are other methods of derivation. The forms of the resultant composite scattering matrices derived by these methods are often different. It is the purpose of this note to reconcile these different forms.

Using the notation in [1] throughout and assuming that the line length L in [1] is zero so that

$$S^{(L)} = \begin{bmatrix} e^{-\gamma_1 L} & 0 & 0 & \\ 0 & e^{-\gamma_2 L} & 0 & \\ 0 & 0 & e^{-\gamma_3 L} & \\ & & & \ddots \end{bmatrix} \quad (1)$$

reduces to the identity matrix I, we have from [1]

$$\begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix} * \begin{pmatrix} S^{33} & S^{34} \\ S^{43} & S^{44} \end{pmatrix} = \begin{pmatrix} S^{AA} & S^{AC} \\ S^{CA} & S^{CC} \end{pmatrix} \quad (2)$$

where the elements S^{ij} are submatrices of infinite order and

$$\begin{pmatrix} S^{AA} & S^{AC} \\ S^{CA} & S^{CC} \end{pmatrix} = \begin{pmatrix} S^{11} + S^{12}U_2S^{33}S^{21} & S^{12}U_2S^{34} \\ S^{43}U_1S^{21} & S^{44} + S^{43}U_1S^{22}S^{34} \end{pmatrix} \quad (3)$$

with

$$U_1 = (I - S^{22}S^{33})^{-1} \quad (4a)$$

$$U_2 = (I - S^{33}S^{22})^{-1}. \quad (4b)$$

The $*$ in (2) is used to denote the cascading process and is, of course, the "star product" of Redheffer [2], [3], [6] originally derived for cascaded $2n$-ports (see Fig. 1). Redheffer's composite scattering matrix was put in the form [7] (still using the notation in [1]):

$$\begin{pmatrix} S^{AA} & S^{AC} \\ S^{CA} & S^{CC} \end{pmatrix} = \begin{pmatrix} S^{11} + S^{12}S^{33}U_1S^{21} & S^{12}U_2S^{34} \\ S^{43}U_1S^{21} & S^{44} + S^{43}S^{22}U_2S^{34} \end{pmatrix}. \quad (5)$$

The off-diagonal elements of (5) are the same as in (3), but the diagonal elements are not. Using the identity

$$A(I - BA)^{-1} = (I - AB)^{-1}A \quad (6)$$

we see that the second term of the upper left element in (5) is

$$S^{12}S^{33}U_1S^{21} = S^{12}S^{33}(I - S^{22}S^{33})^{-1}S^{21} = S^{12}(I - S^{33}S^{22})^{-1}S^{33}S^{21} \quad (7)$$

and the second term in the lower right element in (5) is

$$S^{43}S^{22}U_2S^{34} = S^{43}S^{22}(I - S^{33}S^{22})^{-1}S^{34} = S^{43}(I - S^{22}S^{33})^{-1}S^{22}S^{34}. \quad (8)$$

Equations (7) and (8) can now be used in (5); thus the form in (5) is shown to be equivalent to that in (3).

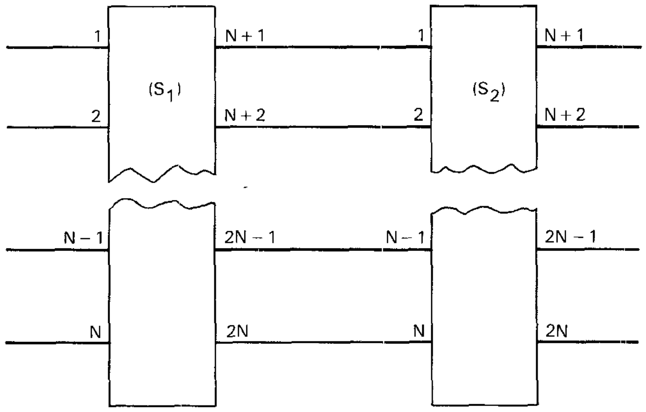

Fig. 1. Cascading two $2n$-ports with scattering matrices S_1 and S_2 (S_1 and S_2 as in [1]).

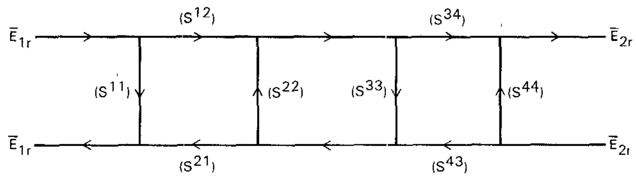

Fig. 2. Signal flow graph for cascaded $2n$-ports of Fig. 1.

A third form of the generalized composite scattering matrix can be found using a signal flow graph analysis. The signal flow graph is readily solved by tracing the vector signals through the network, taking care to maintain the proper order of matrix multiplication. Using Fig. 2, if we cascade S_1 and S_2, we obtain

$$S^{AA} = S^{11} + \left[I - S^{12}S^{33}S^{22}(S^{12})^{-1}\right]^{-1}S^{12}S^{33}S^{21} \quad (9a)$$

$$S^{AC} = \left[I - S^{12}S^{33}S^{22}(S^{12})^{-1}\right]^{-1}S^{12}S^{34} \quad (9b)$$

$$S^{CA} = \left[I - S^{43}S^{22}S^{33}(S^{43})^{-1}\right]^{-1}S^{43}S^{21} \quad (9c)$$

$$S^{CC} = S^{44} + \left[I - S^{43}S^{22}S^{33}(S^{43})^{-1}\right]^{-1}S^{43}S^{22}S^{34}. \quad (9d)$$

Equations (9) can be put in the form of (3) in the following way.

Manuscript received January 4, 1989.
The authors are with the Physics Division, Research Department, Naval Weapons Center, China Lake, CA 93555-6001.
IEEE Log Number 8928842

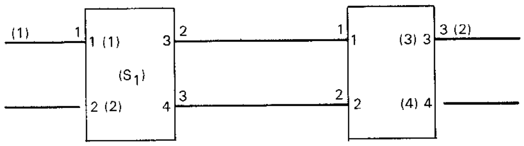

Fig 3. Cascading two three-ports to form a two-port.

Taking the upper left element (9a), we can write it as

$$S^{AA} = S^{11} + \left\{ (S^{12})^{-1} \left[I - S^{12} S^{33} S^{22} (S^{12})^{-1} \right] \right\}^{-1} S^{33} S^{21} \quad (10)$$

$$= S^{11} + \left[(I - S^{33} S^{22})(S^{12})^{-1} \right]^{-1} S^{33} S^{21}$$

$$= S^{11} + S^{12} (I - S^{33} S^{22})^{-1} S^{33} S^{21}. \quad (11)$$

Equation (11) is now the same as the upper left element of (3). S^{AC}, S^{CA}, and S^{CC} in (9) can be transformed to give the remaining elements of (3) in the same way.

While (9) is neither the simplest nor the most numerically convenient of the three forms of the opposite scattering matrix, it is the easiest to derive. Using Fig. 2 and a few signal flow rules, (9) can be written down by inspection.

Although these generalized composite scattering matrix rules apply explicitly to cascaded $2n$-ports, they often can be applied to the cascading of n-port devices (where n is odd) by the introduction of dummy ports with some port renumbering. A simple example is shown in Fig. 3, where two three-ports are cascaded to form a two-port.

Here, dummy ports are added to convert the three-ports into four-ports, and the ports are renumbered to allow application of the composite matrix rules. The numbers exterior to the (black) boxes are the original three-port numbers; the interior numbers are the renumbered four-port numbers. The interior numbers in parentheses are the overall composite four-port numbers, while the exterior numbers in parentheses are the port numbers for the actual resulting two-port.

If l is the port number of a dummy port, then the scattering matrix elements $S_{lj} = S_{jl}$, $j \neq l$, are zero. The reflection coefficients at the dummy ports S_{ll} are arbitrary. Computationally setting them equal to 1 or 0 is convenient. (Note that $S_{lj} = S_{jl} = 0$ will cause infinities in the transmission matrix elements if that approach is attempted.)

Application of the composite scattering matrix rules will of course result in a 4×4 overall scattering matrix; however, it is straightforward to pick out the appropriate scattering coefficients for the desired overall two-port resulting from this particular configuration.

References

[1] T S Chu and T Itoh, "Generalized scattering matrix method for analysis of cascaded and offset microstrip step discontinuities," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34, pp 280–284, Feb 1986

[2] R Redheffer, "Inequalities for a matrix Riccati equation" *J Math. Mech*, vol. 8, pp. 349–367, 1959.

[3] R Redheffer, "On the relation of transmission-line theory to scattering and transfer," *J Math. Phys.*, vol 41, p 1, 1962

[4] R A Speciale, "Progective matrix transformations in microwave network theory, in *IEEE MTT-S Int Microwave Symp. Dig* (Los Angeles, CA), June 15–19, 1981, pp 510–512

[5] J. K. Hunton, "Analysis of microwave measurement techniques by means of signal flow graphs," *IRE Trans Microwave Theory Tech*, vol. MTT-8, pp 206–211, Mar 1960

[6] L. J. Kaplan and D. J. R Stock, "A generalization of the matrix Riccati equation and the "Star" multiplication of Redheffer," *J Math. Mech.*, vol. 11, pp. 927–928, 1962.

[7] D J. R. Stock and L J Kaplan, "A comment on the scattering matrix of cascaded $2n$-ports," *IRE Trans Microwave Theory Tech*, vol. MTT-9, p 454, Sept. 1961.